

Discrete-Time Constrained Portfolio Optimization: Strong Duality Analysis

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Abstract: - We study in this paper the strong duality for discrete-time convex constrained portfolio selection problems when adopting a risk neutral computational approach. In contrast to the continuous-time models, there is no known result of the existence conditions in discrete-time models to ensure the strong duality. Investigating the relationship among the primal problem, the Lagrangian dual and the Pliska's \square dual, we prove in this paper that the strong duality can be always guaranteed for constrained convex portfolio optimization problems in discrete-time models when the constraints are expressed by a set of convex inequalities.

Keywords: Portfolio optimization; incomplete market; utility; investment constraints; duality; martingale approach.

I. INTRODUCTION

We consider in this paper the issue of strong duality for convex inequality constrained portfolio selection problems in a discrete financial model. The continuous version of this problem has been investigated extensively since 1992 and some prominent results have been achieved. Xu and Shreve [3,4] show that convex duality approach succeeds in solving problems with no-short-selling constraint. Cvitanic and Karatzas [2] develop convex duality theory for general convex constrained portfolio optimization problems. As Cvitanic and Karatzas [2] confine admissible policies to be bounded adapted processes that make the wealth process nonnegative, the utility function used in their model, $U(\cdot)$, is defined on $(0, +\infty)$ and satisfies (i) $cU(c)$ is nondecreasing on $(0, \infty)$, and (ii) there exist some $\alpha \in (0, 1)$ and $\beta \in (1, \infty)$ such that $U(\alpha x) \geq \alpha U(x)$, $\beta x \in (0, \infty)$. Cvitanic and Karatzas [2] introduce then a family of unconstrained problems and build up the corresponding dual problem. Finally, they prove the strong duality theorem that the optimal solution of the dual problem also solves the primal problem.

For discrete financial models studied in this paper, we define admissible policies to be general bounded adapted processes and thus define the objective utility on the entire R . Similar to Cvitanic and Karatzas [2], Pliska [1] introduces a family of unconstrained problems for a constrained discrete financial model and gives the strong duality condition under which the optimal solution of the dual problem also solves the primal problem. However, to our knowledge, there is no known result in the literature on the existence condition such that the strong duality condition can be ensured to hold in a discrete-time model as the continuous-time model does.

In this paper, we would like to close the gap between continuous-time models and discrete-time models, and prove that the strong duality condition always holds in discrete-time models when utility function satisfies $cU(c+\alpha) \leq cU(c)$ for $c \in R$ and $\alpha \in (-\infty, 0)$. We build up in Section 2 a discrete-time financial model, and formulate the constrained portfolio selection problem mathematically. We discuss in Section 3 the Pliska's \square dual problem and present the strong duality theorem. We derive in Section 4 the main result of this paper: Theory of guaranteed strong duality. We demonstrate our results via an illustrative example in Section 5 before we conclude our paper in Section 6.

II. MATHEMATICAL FORMULATION

We consider a financial market, consisting of n risky assets and one risk free asset, in which investors make their investment decisions at multiple time instants, $t=0, 1, \dots, T-1$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, P)$ be the filtrated probability space, where $\Omega := \{\omega_1, \dots, \omega_K\}$ is the sample space with K finite samples. Denote the stochastic process of the risky securities' returns as $\boldsymbol{\mu} = \{\boldsymbol{\mu}_t\}_{t=0, \dots, T-1}$, where $\boldsymbol{\mu}_t = (\mu_t(1), \dots, \mu_t(n))'$ is a random vector, and the bond return process as $\mathbf{r} = \{r_t\}_{t=0, \dots, T-1}$, where r_t is a deterministic scalar. Denote $\mathbf{R} = \{R_t\}_{t=0, \dots, T-1}$ as the extra return process with $R_t = (R_t(1), \dots, R_t(n))'$ and $R_t(i) = \mu_t(i) - r_t$.

Furthermore, we introduce two assumptions on the financial market, which guarantee the completeness of the market.

Assumption 2.1.(i) At any time t , there exist $m_t := (n+1)^t$ elements $A_t^1, \dots, A_t^{m_t}$, such that $A_t^1 \cup \dots \cup A_t^{m_t} = \Omega$, $A_t^i \cap A_t^j = \emptyset, \forall i \neq j$, and $F_t = \sigma(A_t^1, \dots, A_t^{m_t})$; (ii) $\forall i \leq m_t$, $A_{t+1}^{(i-1)(n+1)+j} \subset A_t^i$ for $j=1, \dots, n+1$; (iii) The assets' return matrix

$$\begin{pmatrix} r_t & \dots & r_t \\ \mu_t(A_t^{(i-1)(n+1)+1}) & \dots & \mu_t(A_t^{i(n+1)}) \end{pmatrix}_{(n+1) \times (n+1)}$$

is full rank for any $A_t^i \in F_t$.

Assumption 2.2. The financial market is arbitrage-free.

An investor with initial wealth v would like to invest her wealth in the market. Denote her self-financing trading strategies as $\Pi = \{\pi_t\}_{t=0,1,\dots,T-1}$, where $\pi_t = (\pi_t(1), \dots, \pi_t(n))'$ with $\pi_t(i)$ being the dollar amount invested in i th risky security at time t . Let V_t be the portfolio value at time t . The dollar amount invested in the bond at time t is then $V_t - \sum_{i=1}^n \pi_t(i)$. Therefore, the wealth process satisfies

$$V_{t+1} = V_t r_t + R_t' \pi_t. \tag{1}$$

We assume in this paper that, when there is no constraint on trading strategies, the market governed by the stochastic difference equation described above,

$$\begin{cases} V_{t+1} = V_t r_t + R_t' \pi_t; \\ \pi_t \in \square^n, t = 0, 1, \dots, T-1; \\ V_0 = v, \end{cases}$$

satisfies both Assumptions 2.1 and 2.2, thus being a complete market.

The subject we study in this paper is a constrained portfolio selection problem. Under the condition that π_t is constrained in a convex set \mathbf{K}_t , the investor pursues her investment by maximizing her expected utility of the terminal wealth, $U(\cdot): \square \times \Omega \rightarrow \square$, where $U(\square, \square)$ is assumed to be differentiable, strictly increasing and concave for each $\square \square \square$. For example, $\mathbf{K}_t = \{\pi_t \in \square^n; \pi_t(i) \geq 0, \forall i = 1, \dots, n\}$ when short selling is prohibited. In summary, the mathematical model of the investor's constrained portfolio selection problem is posted as follows,

$$(P) \begin{cases} \max_{\pi} & E[U(V_T)] \\ \text{s.t.} & V_{t+1} = V_t r_t + (R_t)' \pi_t; \\ & \pi_t \in \mathbf{K}_t \subseteq \square^n, t = 0, 1, \dots, T-1; \\ & V_0 = v. \end{cases}$$

If \mathbf{K}_t is a subset of \square^n , problem (P) is a portfolio selection problem in an incomplete market, as some contingent claims can not be hedged by any admissible portfolios due to the constraints.

III. RISK NEUTRAL COMPUTATIONAL APPROACH

Following Pliska [1], we define the support function of $-\mathbf{K}_t$ as

$$\delta(\kappa_t) \equiv \sup_{\pi_t \in \mathbf{K}_t} -(\pi_t)' \kappa_t.$$

The effective domain of $\delta(\kappa_t)$ is then given by

$$\tilde{\mathbf{K}}_t = \{\kappa_t \in \square^n; \delta(\kappa_t) < \infty\}.$$

We introduce the predictable stochastic process $\kappa := \{\kappa_t; t = 0, 1, \dots, T-1, \kappa_t \in \tilde{\mathbf{K}}_t\}$. Let \mathbf{D} be the set of all such processes of \square . For each $\square \in \mathbf{D}$, we construct an auxiliary market \mathbf{M}_κ with the following modified returns,

$$\begin{aligned} r_t^\kappa &= r_t + \frac{\delta(\kappa_t)}{V_t^\kappa}, \\ \mu_t^\kappa(i) &= \mu_t(i) + \frac{\delta(\kappa_t)}{V_t^\kappa} + \kappa_t(i), i = 1, \dots, n, \\ V_{t+1}^\kappa &= V_t^\kappa r_t^\kappa + (R_t^\kappa)' \pi_t, \end{aligned}$$

where $R_t^\kappa = (R_t^\kappa(1), \dots, R_t^\kappa(n))'$ with $R_t^\kappa(i) = \mu_t^\kappa(i) - r_t^\kappa = R_t(i) + \kappa_t(i)$.

When $V_t^\kappa = 0$, we let $r_t^\kappa = r_t$ and $\mu_t^\kappa(i) = \mu_t(i) + \kappa_t(i)$. Notice that $\frac{\pi_t}{V_t^\kappa}$ is the proportional trading strategy adopted in Pliska [1].

The first step in the risk neutral computational approach (see [1]) is to embed the primal constrained portfolio selection problem (P) into a family of unconstrained portfolio selection problems in \mathbf{M}_κ ,

$$(P^\kappa) \begin{cases} \max_{\pi} & E[U(V_T^\kappa)] \\ \text{s.t.} & V_{t+1}^\kappa = V_t^\kappa r_t^\kappa + (R_t^\kappa)' \pi_t; \\ & \pi_t \in \square^n; t = 0, 1, \dots, T-1; \\ & V_0 = v. \end{cases}$$

Note that Assumption 2.1 still holds in the auxiliary market \mathbf{M}_κ , as the return matrix in the auxiliary market \mathbf{M}_κ is obtained by performing some elementary transformations on the return matrix of the original market, due to the predictability of process \square .

Thus, problem (P^κ) for given $\square \in \mathbf{D}$ can be still efficiently solved by using the martingale-like approach in [1].

It is easy to see that, for $\pi_t \in \square^n, t = 0, 1, \dots, T-1$,

$$\begin{aligned} V_T^\kappa &= V_{T-1}^\kappa r_{T-1} + (R_{T-1}^\kappa)' \pi_{T-1} + [\delta(\kappa_{T-1}) + (\pi_{T-1})' \kappa_{T-1}] \\ &\geq V_{T-1}^\kappa r_{T-1} + (R_{T-1}^\kappa)' \pi_{T-1} \\ &= [V_{T-2}^\kappa r_{T-2} + (R_{T-2}^\kappa)' \pi_{T-2}] r_{T-1} + (R_{T-1}^\kappa)' \pi_{T-1} \\ &\vdots \\ &\geq v \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (R_t)'\pi_t \prod_{i=t+1}^{T-1} r_i \\ &= V_T. \end{aligned}$$

Due to the increasing property of the utility function, we can get the following weak duality.

Proposition 3.1. Weak duality Let $J(v)$ be the optimal value of primal problem (P) and $J^\kappa(v)$ be the optimal value of problem (P^κ) . Then

$$J(v) \leq J^\kappa(v), \forall \kappa \in \mathbf{D}.$$

As for any $\square \in \mathbf{D}$, $J^\kappa(v)$ offers an upper bound for $J(v)$, the second step in the risk neutral computational approach [1] is to find the tightest upper bound by solving the following dual problem,

$$(D) \quad \kappa^* = \operatorname{argmin}_{\kappa \in \mathbf{D}} J^\kappa(v),$$

such that, hopefully, the optimal solution to the unconstrained problem in the market \mathbf{M}_{κ^*} will turn out to be the optimal solution to the constrained problem in the primal constrained market and the corresponding optimal

objective values will coincide, i.e., $J^{\kappa^*}(v) = J(v)$. We call the dual problem (D) in this paper as the Pliska's dual of (P).

Proposition 3.2. Strong Duality (see [1]) Suppose that for some $\hat{\kappa} \in \mathbf{D}$, the optimal trading strategy of $(P^{\hat{\kappa}})$, $\hat{\pi}$, satisfies

- (a) $\hat{\pi}_t \in \mathbf{K}_t$,
- (b) $\delta(\hat{\kappa}_t) + \hat{\pi}_t' \hat{\kappa}_t = 0$.

Then $\hat{\pi}$ is optimal for the primal constrained portfolio selection problem (P), and $J(v) = J^{\hat{\kappa}}(v) \leq J^{\kappa}(v)$ for all $\kappa \in \mathbf{D}$.

A crucial question is the existence guarantee of such a $\hat{\kappa}$ for achieving strong duality. Pliska states the following in [1]: *The obvious candidate for such a $\hat{\kappa}$ is κ^* , the solution of the dual problem (D). After computing κ^* , you then check whether π^* , the optimal trading strategy for (P^{κ^*}) , satisfies conditions (a) and (b) in Proposition 3.2. If both conditions are satisfied, then π^* will be optimal for the primal constrained portfolio selection problem (P).* However, as emphasized in [1], there is no known result to guarantee such an existence.

The main purpose of this paper is to present a guaranteed strong duality result when convex set \mathbf{K}_t is specified by a set of convex inequalities.

IV. GUARANTEED STRONG DUALITY

Let us consider problem (P), where feasible convex set \mathbf{K}_t is specified by a set of convex inequalities,

$$\mathbf{K}_t = \{\pi; G_t(\pi) \leq b_t\}, \tag{2}$$

where $G_t := (G_t^1, G_t^2, \dots, G_t^{d_t})'$ with $G_t^i \in \mathbb{R}^2$ being a second order continuous differentiable convex function, $i = 1, \dots, k$, and b_t is a d_t dimensional vector.

As we know, the primal problem (P) can be tackled either as a stochastic control problem, where the trading strategy at time t , π_t , is a \mathcal{F}_t -measurable stochastic random vector, or as a static optimization problem, where all the realizations of π_t are considered separately based on our discrete financial model. In the latter case, the objective function in (P) can be reformulated as follows,

$$\sum_{\omega \in \Omega} P(\omega) U \left(v \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (R_t(\omega))' \pi_t(\omega) \prod_{i=t+1}^{T-1} r_i \right).$$

Notice that $\pi_t(\omega) = \pi_t(A_t^i)$ if $\omega \in A_t^i$, due to the tree structure of the market. The decision vectors are $\pi_t(A_t^i)$ for $t=0, 1, \dots, T-1$ and $i = 1, \dots, (n+1)^t$. When we deal with the primal problem (P) as a static one, we first formulate its Lagrangian dual problem.

Given $\mathbf{K}_t = \{\pi; G_t(\pi) \leq b_t\}$, the Lagrangian dual of problem (P) is given as follows,

$$\begin{aligned} (D_L) \quad & \min_{\lambda} \max_{\pi} \sum_{t=0}^{T-1} \sum_i^{(n+1)^t} (\lambda_t(A_t^i))' [b_t - G_t(\pi_t(A_t^i))] \\ & + \sum_{\omega \in \Omega} P(\omega) U \left(v \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (R_t(\omega))' \pi_t(\omega) \prod_{i=t+1}^{T-1} r_i \right), \end{aligned}$$

where $\lambda = \{\lambda_t\}_{t=0,1,\dots,T-1}$ is a nonnegative adapted process. As problem (P) is convex, there is no duality gap between problems (P) and (D_L) from the strong duality theorem. Furthermore, a process pair (π^*, λ^*) satisfying the first order condition,

$$E[1_{A_t^i} U' (v \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (R_t)' \pi_t^*(A_t^i) \prod_{i=t+1}^{T-1} r_i) R_t \prod_{i=t+1}^{T-1} r_i] - (\lambda_t^*(A_t^i))' G_t'(\pi_t^*(A_t^i)) = 0, \tag{3}$$

$$(\lambda_t^*(A_t^i))'(b_t - G_t(\pi_t^*(A_t^i))) = 0, \tag{4}$$

for $i = 1, \dots, (n+1)^t$ and $t=0, 1, \dots, T-1$, where

$$G_t'(\pi_t) = \begin{pmatrix} (\nabla G_t^1)' \\ \vdots \\ (\nabla G_t^k)' \end{pmatrix},$$

solves both the primal and the Lagrangian dual problems.

Theorem 4.1. Assume that the concave utility function further satisfies

$$cU'(c+\square) < \square, \tag{5}$$

for all $c \in \mathbb{R}$ and $\square \in (\mathbb{R}^+, +)$. For the process pair (π^*, λ^*) specified in (3) and (4), respectively, let

$$\kappa_t^*(A_t^i) = -\frac{(G_t'(\pi_t^*(A_t^i)))' \lambda_t^*(A_t^i)}{\Phi_t(A_t^i)}, \tag{6}$$

for $i = 1, \dots, (n+1)^t$ and $t=0, 1, \dots, T-1$, where

$$\Phi_t(A_t^i) := E[1_{A_t^i} U'(v \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (R_t)^? \pi_t^*(A_t^i) \prod_{i=t+1}^{T-1} r_t) \prod_{i=t+1}^{T-1} r_i].$$

Then $\pi_t^* \in \mathbf{K}_t$, $\kappa_t^* \in \mathbf{K}_t$, $\{\pi_t^*\}$ solves (P^{κ^*}) and

$$\delta(\kappa_t^*) + (\kappa_t^*)' \pi_t^* = 0, \tag{7}$$

for $t=0, 1, \dots, T-1$.

Proof. The conclusion of $\pi_t^* \in \mathbf{K}_t$ is due to the strong duality between the primal problem (P) and the Lagrangian dual (D_L) . The following is clear from (5),

$$(\pi_t^*(A_t^i))'(G_t'(\pi_t^*(A_t^i)))' \lambda_t^*(A_t^i) = E[1_{A_t^i} U'(v \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (R_t)^? \pi_t^* \prod_{i=t+1}^{T-1} r_t) (\pi_t^*)' R_t \prod_{i=t+1}^{T-1} r_i] < \infty. \tag{8}$$

As $G_t^i(\pi_t) = G_t^i(\pi_t^*) + (\nabla G_t^i(\pi_t^*))'(\pi_t - \pi_t^*) + 0.5(\pi_t - \pi_t^*)' HG_t^i(\xi)(\pi_t - \pi_t^*) \leq b_t^i$ for some \square between π_t and π_t^* , $i=1, \dots, k$, where $HG_t^i(\xi)$ is the Hessian matrix, we have

$$\nabla G_t^i(\pi_t^*) \pi_t \leq b_t^i - G_t^i(\pi_t^*) + \nabla G_t^i(\pi_t^*) \pi_t^* - 0.5(\pi_t - \pi_t^*)' HG_t^i(\xi)(\pi_t - \pi_t^*).$$

Since $(\lambda_t^*)'(b_t - G_t(\pi_t^*)) = 0$, we further have

$$(\lambda_t^*)' G_t'(\pi_t^*) \pi_t \leq (\lambda_t^*)' G_t'(\pi_t^*) \pi_t^* - 0.5(\lambda_t^*)' \mathbf{H}_t \leq (\lambda_t^*)' G_t'(\pi_t^*) \pi_t^* < \infty, \tag{9}$$

where

$$\begin{aligned} \mathbf{H}_t &= (H_t^1, \dots, H_t^k)', \\ H_t^i &= (\pi_t - \pi_t^*)' HG_t^i(\xi)(\pi_t - \pi_t^*) \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \delta(\kappa_t^*(A_t^i)) &= \sup_{\pi_t \in \mathbf{K}_t} -(\pi_t(A_t^i))' \kappa_t^*(A_t^i) = \\ \sup_{\pi_t \in \mathbf{K}_t} &\frac{(\pi_t(A_t^i))'(G_t'(\pi_t^*(A_t^i)))' \lambda_t^*(A_t^i), \Phi_t(A_t^i)}{\Phi_t(A_t^i)} < \infty, \end{aligned}$$

which implies $\kappa_t^* \in \tilde{\mathbf{K}}_t$.

The following is clear from (9),

$$\delta_t(\kappa_t^*) + (\kappa_t^*)' \pi_t^* = \sup_{\pi_t \in \mathbf{K}_t} \{-(\pi_t)' \kappa_t^*\} + (\kappa_t^*)' \pi_t^*$$

$$\begin{aligned}
 &= \sup_{\pi_t \in \mathbf{K}_t} \frac{(\lambda_t^*)' G_t'(\pi_t^*) \pi_t}{\Phi_t(A_t^i)} - \frac{(\pi_t^*)' (G_t'(\pi_t^*))' \lambda_t^*}{\Phi_t} (A_t^i) \\
 &= \sup_{\pi_t \in \mathbf{K}_t} \frac{(\lambda_t^*)' G_t'(\pi_t^*) \pi_t - (\lambda_t^*)' G_t'(\pi_t^*) \pi_t^*}{\Phi_t(A_t^i)} \\
 &= 0.
 \end{aligned}$$

Furthermore, we can check that π^* satisfies the following optimality condition of problem (P^{K^*}) ,

$$\Psi_t(A_t^i) := E[1_{A_t^i} (R_t + \kappa_t^*) \prod_{i=t+1}^T r_i^{K^*} \times U'(v \prod_{t=0}^{T-1} r_t^{K^*} + \sum_{t=0}^{T-1} (\pi_t^*)' (R_t + \kappa_t^*) \prod_{i=t+1}^{T-1} r_i^{K^*})] = 0.$$

Actually, we can derive the following equation by (7),

$$V_T^{K^*} = v \prod_{t=0}^{T-1} r_t^{K^*} + \sum_{t=0}^{T-1} (\pi_t^*)' (R_t + \kappa_t^*) \prod_{i=t+1}^{T-1} r_i^{K^*} = v \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (\pi_t^*)' R_t \prod_{i=t+1}^{T-1} r_i = V_T.$$

Therefore, the following equation can be derived from (3) and (6),

$$\begin{aligned}
 \Psi_t(A_t^i) &= E[1_{A_t^i} U'(V_T^{K^*}) (R_t + \kappa_t^*) \prod_{i=t+1}^{T-1} r_i^{K^*}] = E[1_{A_t^i} U'(V_T) (R_t + \kappa_t^*) \prod_{i=t+1}^{T-1} r_i^{K^*}] \\
 &= E[1_{A_t^i} U'(V_T) R_t \prod_{i=t+1}^{T-1} r_i^{K^*}] + E[1_{A_t^i} U'(V_T) \kappa_t^* \prod_{i=t+1}^{T-1} r_i^{K^*}] \\
 &= (G_t'(\pi_t^*))' \lambda_t^* \prod_{i=t+1}^{T-1} \frac{r_i^{K^*}}{r_i} - (G_t'(\pi_t^*))' \lambda_t^* \prod_{i=t+1}^{T-1} \frac{r_i^{K^*}}{r_i} \\
 &= 0.
 \end{aligned}$$

Hence, π^* solves (P^{K^*}) .

Remark 4.1 If the optimal (κ^*, π^*) for the Pliska's \square dual problem can be derived, and matrix $G_t'(\pi_t^*)$ are nonsingular, then the unique optimal Lagrangian multiplier λ^* for the Lagrangian dual problem can be found as,

$$\lambda_t^*(A_t^i) = -(G_t'(\pi_t^*(A_t^i)))'. \tag{10}$$

When some matrix $G_t'(\pi_t^*)$ are singular or even not square, the optimal Lagrangian can not be uniquely determined by the relationship (6).

5. Illustrative Examples

Example 1. We study now a single-period investment example with one risky asset and no short-selling constraint to illustrate the relationship of the primal problem, the Lagrangian dual problem and the Pliska's \square dual problem. The primal problem (P_0) is given as follows,

$$(P_0) \begin{cases} \max_{\pi} & E[U(V_1)] \\ \text{s.t.} & V_1 = vr + R\pi; \\ & \pi \geq 0. \end{cases}$$

While the corresponding Lagrangian dual is

$$\min_{\lambda \geq 0} f(\lambda) := \max_{\pi} E[U(vr + R\pi)] - \lambda\pi,$$

the associated Pliska's \square dual is

$$\min_{\kappa \geq 0} g(\kappa) := \max_{\pi} E[U(v(r + \kappa + \frac{\delta(\kappa)}{v}) + R\pi + \kappa\pi)].$$

We can depict the objective functions of the two dual problems in the same figure. Two situations may occur according to different returns of risky security, R . Figure 1 represents the situation when the optimal Lagrangian multiplier λ^* is bigger than zero. In such a situation, λ^* and the corresponding dual parameter κ^* may not be equal. However, both the optimal objective values are equal to the optimal objective value of primal problem.

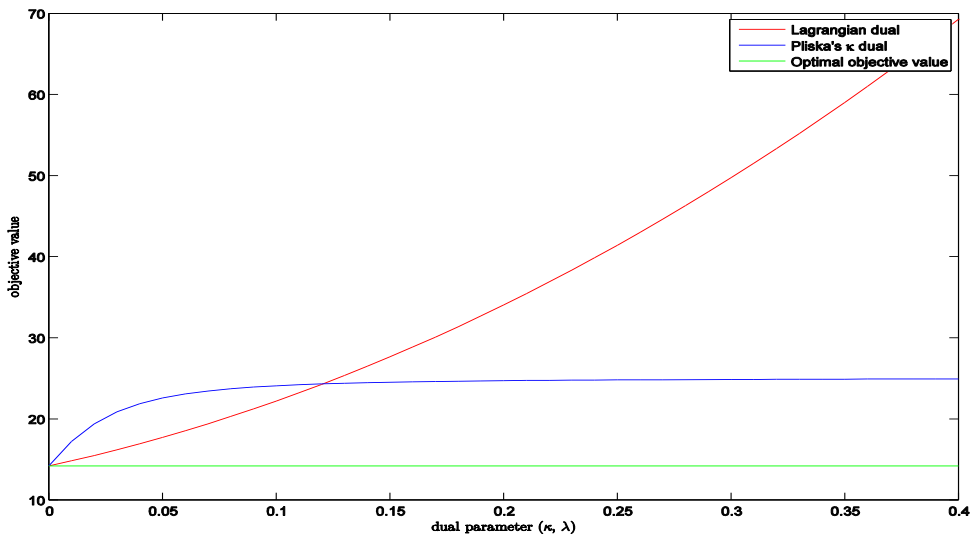


Figure 1: Situation with optimal parameters bigger than 0

Figure 2 illustrates the situation when the optimal Lagrangian multiplier λ^* equals to zero. In such a situation, both λ^* and the corresponding parameter κ^* are equal to zero, and the optimal objective values of the lagrangian dual and the Pliska's κ dual are still equal to the optimal objective value of the primal problem.

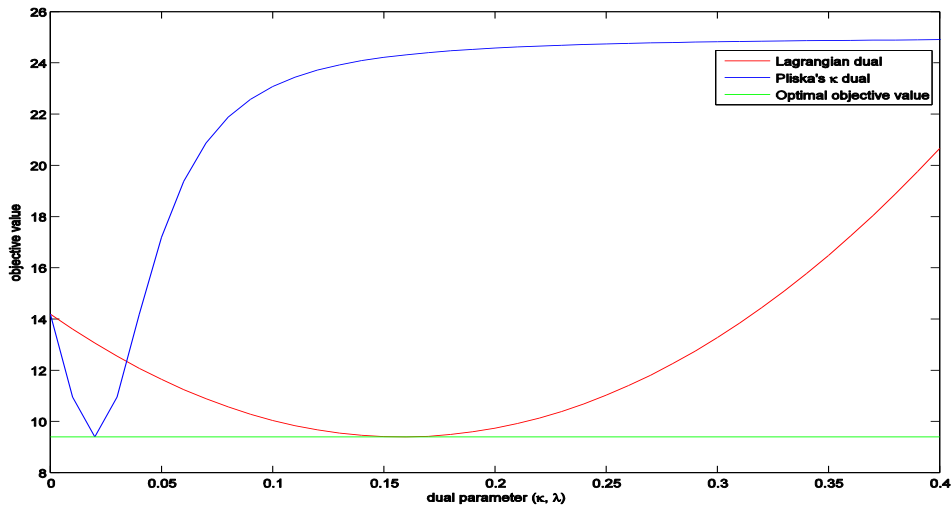


Figure 2: Situation with optimal parameters equal to 0

Example 2. Consider Example 5.11 in [1] which is a two-period problem with short-selling prohibited, the bond return rate given as $r_1 = r_0 = 1$ and the price process for the single risky asset and the probability measure specified as follows:

ω	$\mu_0(\omega)$	$\mu_1(\omega)$	$P(\omega)$
ω_1	8/5	9/8	1/4
ω_2	8/5	6/8	1/4
ω_3	4/5	6/4	1/4
ω_4	4/5	3/4	1/4

An investor with a log utility function, $U(V)=\ln(V)$, enters the financial market with initial wealth $v=1$. The optimal dual parameter λ , trading strategies and wealth process have been derived in [1]. More specifically, the corresponding optimal process κ^* is

$$\kappa_0^* = 0, \kappa_1^*({\omega_1, \omega_2}) = \frac{1}{16}, \kappa_1^*({\omega_3, \omega_4}) = 0,$$

and the optimal trading strategies are

$$\pi_0^* = \frac{5}{3}, \pi_1^*({\omega_1, \omega_2}) = 0, \pi_1^*({\omega_3, \omega_4}) = \frac{2}{3},$$

yielding the corresponding optimal terminal wealth process as

$$V_1({\omega_1, \omega_2}) = 2, V_1({\omega_3, \omega_4}) = \frac{2}{3},$$

$$V_2(\omega_1) = 2, V_2(\omega_2) = 2, V_2(\omega_3) = 1, V_2(\omega_4) = \frac{1}{2}.$$

Solving the Lagrangian dual problem,

$$\min_{\lambda} \max_{\pi} E(\ln(V_2)) - \lambda_0 \pi_0 - \lambda_1({\omega_1, \omega_2}) \pi_1({\omega_1, \omega_2}) - \lambda_1({\omega_3, \omega_4}) \pi_1({\omega_3, \omega_4}),$$

gives rise the optimal Lagrangian parameter process, λ^* ,

$$\lambda_0^* = 0, \lambda_1^*({\omega_1, \omega_2}) = \frac{1}{64}, \lambda_1^*({\omega_3, \omega_4}) = 0.$$

It can be verified that the optimal trading strategies derived from the Lagrangian dual and Pliska's λ dual are exactly the same, and, furthermore, (6) holds.

V. CONCLUSION

By identifying the relationship between the Lagrangian dual and Pliska's λ dual for the constrained portfolio selection problem, we have derived in this paper a guaranteed strong duality result for a class of discrete-time constrained convex portfolio selection problems. More specifically, we ensure the existence of an optimal λ in the strong duality conditions of [1] to guarantee the success of the risk neutral computational approach.

The authors declare that there is no conflict of interests.

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